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ON THE INFLUENCE OF TEMPERATURE AND VISCOSITY FLUCTUATIONS ON INTERFACIAL INSTABILITY

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ABSTRACT

Recently, Pinarbasi and Liakopoulos [1] investigated the effect of variable viscosity on the interfacial stability of two-layer, plane, Poiseuille flow. Temperature and viscosity fluctuations were neglected based on the observation that their effect on the stability of single-layer, Poiseuille flow is very small [2,3]. It is shown in the present work that temperature and viscosity fluctuations, under some conditions, have a significant effect on the interfacial mode of instability, especially for large Prandtl numbers.

Introduction

In two recent studies [2,3], the effect of temperature dependent viscosity on the *shear mode* of instability of *single-layer*, plane, Poiseuille flow was investigated. Both studies report that although the dependence of viscosity on temperature affects considerably the value of the critical Reynolds number and the location of neutral stability curves, temperature and viscosity fluctuations have no appreciable effect on linear instability.

Interfacial instabilities in Couette or Poiseuille flow of two immiscible fluids have been investigated by various authors under *isothermal* conditions. Yih's [4] longwave asymptotic analysis of two-layer flow showed that viscosity stratification alone can lead to instability. Recent studies on interfacial instability (e. g., Yiantsios and Higgins [5], and Su and Khomami [6]) have utilized numerical techniques to investigate the stability of the interface to disturbances of arbitrary wavelength.

The study of interfacial instabilities in *nonisothermal*, pressure-driven, two-layer flows is a largely unexplored area despite its direct relevance to the analysis of coextrusion flows. Anturkar *et al.* [7] studied the interfacial instability of two-layer flow incorporating the temperature dependence of material properties. However, heat transfer across the layers was neglected and the standard isothermal stability equations were solved. In a recent study, Pinarbasi and Liakopoulos [1] considered nonisothermal Poiseuille flow of two superposed Newtonian fluids with temperature dependent viscosity. Viscosity and temperature fluctuations were neglected in their analysis. The aim of this work is to investigate the effect of temperature and viscosity fluctuations on the interfacial stability of confined, two-layer, pressure-driven flows.

Governing Equations

We examine the stability of the interface between two immiscible, incompressible, Newtonian liquids flowing steadily between two parallel plates under nonisothermal conditions. The dimensionless governing equations take the form

$$\nabla \cdot \vec{v_k} = 0 , \tag{1}$$

$$\frac{\partial \vec{v_k}}{\partial t} + \vec{v_k} \cdot \nabla \vec{v_k} = - \frac{\nabla p_k}{r_k} + \frac{1}{r_k Re} \nabla \cdot (\mu_k \dot{\gamma}_k) , \tag{2}$$

$$\frac{\partial T_k}{\partial t} + \vec{v_k} \cdot \nabla T_k = \frac{m_k}{r_k Re Pr_k} \nabla^2 T_k , \tag{3}$$

where the viscosity-temperature relation is of the form $\bar{\mu}_k = C_k \bar{\mu}_{o_k} \exp(\bar{d}_k/\bar{T}_k)$ leading to a non-dimensional viscosity law

$$\mu_k = \frac{\bar{\mu}_k}{\bar{\mu}_{o_1}} = C_k m_k \exp\left(\frac{D_k}{T_k}\right). \tag{4}$$

In the equations above, $k = 1, 2$ denotes the layer number (no summation over k), $\dot{\gamma}$ is the rate-of-strain tensor, $\vec{v_k} = (\bar{u}_k, \bar{v}_k)/\bar{U}_o$ is the velocity vector, $p_k = \bar{p}_k/(\bar{\rho}_1 \bar{U}_o^2)$ denotes the pressure, $T_k = \bar{T}_k/\bar{d}_1$ is the temperature, $Re = \bar{\rho}_1 \bar{U}_o \bar{l}_1/\mu_{o_1}$ is the Reynolds number, $Pr_k = \mu_{o_k} c_{p_k}/k_k$ denotes the Prandtl number for the k-th layer, $m_k = \bar{\mu}_{o_k}/\bar{\mu}_{o_1}$ so that m_2 is the viscosity ratio at the reference temperature, $r_k = \bar{\rho}_k/\bar{\rho}_1$ so that r_2 is the density ratio, and $D_k = \bar{d}_k/\bar{d}_1$. C_1 and C_2 are dimensionless constants obtained from temperature-viscosity curves. Properties of the upper layer are used as reference values in the formation of all ratios and the Reynolds number is based on the interface velocity, $\bar{U}_o$, and the upper layer thickness, $\bar{l}_1$.

We assume that, in the undisturbed state, the flow is fully developed and that the temperature field is independent of the streamwise coordinate x. Then, the primary (base) flow can be expressed in terms of exponential integrals (see Eqs. 11 and 12 in Ref. 1) and the base temperature distribution is necessarily a piecewise linear function determined by the imposed wall temperature difference $\Delta\overline{T}$, the layer thickness ratio $\varepsilon = \overline{l}_2/\overline{l}_1$, the thermal conductivity ratio $K_2 = \overline{k}_2/\overline{k}_1$ and the lower (cold) wall temperature $\overline{T}_c$. The temperature of the cold wall is chosen as the reference temperature in viscosity calculations.

Linear Stability Analysis

Due to the complexity of the problem, we restrict our analysis to two-dimensional disturbances although there is no Squire's theorem for two-layer Poiseuille flow (Yiantsios and Higgins [5]). Following standard linear hydrodynamic stability methodology (Drazin and Reid [8], Simpkins and Liakopoulos [9]), we derive the following stability equations governing the evolution of small, two-dimensional disturbances

$$i\alpha Re r_k\{(u_{b_k} - c)(\phi_k'' - \alpha^2\phi_k) - u_{b_k}''\phi_k\} = \mu_{b_k}(\phi_k^{IV} - 2\alpha^2\phi_k'' + \alpha^4\phi_k)$$

$$+ 2\mu_{b_k}'(\phi_k''' - \alpha^2\phi_k') + \mu_{b_k}''(\phi_k'' + \alpha^2\phi_k) + u_{b_k}'(\Lambda_k'' + \alpha^2\Lambda_k) + 2u_{b_k}''\Lambda_k' + u_{b_k}'''\Lambda_k, \qquad k = 1, 2. \quad (5)$$

and

$$(i\alpha r_k Re Pr_k/m_k)\{(u_{b_k} - c)\theta_k - \phi_k T_{b_k}'\} = \theta_k'' - \alpha^2\theta_k\,, \qquad k = 1, 2. \quad (6)$$

Here, primes denote differentiation with respect to y (coordinate normal to the channel walls), subscript b denotes base state quantities, α is the wavenumber, c is the complex disturbance velocity, and ϕ_k, θ_k and Λ_k are the amplitudes of velocity, temperature and viscosity perturbations, respectively. The amplitude of the perturbation viscosity Λ_k in Eq. (5) can be eliminated through the relationship

$$\Lambda_k(y) = \beta_k(y)\,\theta_k(y)\,, \quad (7)$$

where

$$\beta_k(y) = -\frac{C_k m_k D_k}{T_{b_k}^2}\, e^{\frac{D_k}{T_{b_k}}}. \quad (8)$$

The associated linearized boundary conditions are

• no slip at the channel walls

$$\phi_1 = \phi_1' = 0 \qquad \text{at } y = 1,$$

$$\phi_2 = \phi_2' = 0 \qquad \text{at } y = -\varepsilon \quad (9a)$$

• continuity of velocity at the interface

$$\phi_1 = \phi_2\,; \qquad \phi_1' - \phi_2' = \frac{\phi_1}{c - u_{b_1}}\{u_{b_2}' - u_{b_1}'\} \qquad \text{at } y = 0 \tag{9b}$$

• continuity of shear stress at the interface

$$\mu_{b_1}(\phi_1'' + \alpha^2\phi_1) + u_{b_1}'\beta_1\theta_1 = \mu_{b_2}(\phi_2'' + \alpha^2\phi_2) + u_{b_2}'\beta_2\theta_2 \qquad \text{at } y = 0 \tag{9c}$$

• continuity of normal stress at the interface

$$\mu_{b_2}\phi_2''' + \mu_{b_2}'\phi_2'' - (3\mu_{b_2}\alpha^2 + i\alpha Rer_2 u_{b_2})\phi_2' + (\mu_{b_2}'\alpha^2 + i\alpha Rer_2 u_{b_2}')\phi_2$$

$$- \mu_{b_1}\phi_1''' - \mu_{b_1}'\phi_1'' + (3\mu_{b_1}\alpha^2 + i\alpha Re u_{b_1})\phi_1' - (\mu_{b_1}'\alpha^2 + i\alpha Re u_{b_1}')\phi_1$$

$$- c[i\alpha Re\phi_1' - i\alpha Rer_2\phi_2'] = i\alpha Re(F + \alpha^2 S)\frac{\phi_1}{c - u_{b_1}} \qquad \text{at } y = 0 \tag{9d}$$

• constant temperature at the channel walls

$$\theta_1 = 0 \quad \text{at } y = 1, \qquad \theta_2 = 0 \quad \text{at } y = -\varepsilon \tag{9e}$$

• continuity of temperature at the interface

$$\frac{\phi_1}{c - u_{b_1}}\{T_{b_1}' - T_{b_2}'\} = \theta_2 - \theta_1 \qquad \text{at } y = 0 \tag{9f}$$

• continuity of heat flux at the interface

$$\theta_1' = K_2\theta_2' \qquad \text{at } y = 0\,. \tag{9g}$$

In the equations above, $S = \bar{\sigma}/(\bar{\rho}_1\bar{U}_o^2\bar{l}_1)$ and $F = (\bar{\rho}_2 - \bar{\rho}_1)\bar{g}\bar{l}_1/(\bar{\rho}_1\bar{U}_o^2)$ are dimensionless groups accounting for the effects of interfacial tension and gravitational acceleration. Eqs. (9a)-(9d) correspond to Eqs. (18a)-(18d) in Ref. 1, but now Eq. (9c) contains viscosity fluctuation terms. Eqs. (9e) through (9g) are four additional boundary conditions that appear here due to the inclusion of temperature fluctuations in the formulation. It should also be noted that the true interfacial conditions must be imposed at the deflected interface [4]. Consequently, the linearized interfacial conditions (9b, 9c, 9d, 9f, 9g) were derived by considering Taylor series expansions about the undisturbed interface ($y = 0$) and neglecting nonlinear terms. The eigenvalue problem, Eqs. (5)-(9), was solved numerically by a Chebyshev spectral method. Ref. 1 contains further details on the numerical solution technique. Results of various *limit cases* [1,6, 10] were reproduced with excellent agreement. Table 1 presents a comparison for isothermal flow.

TABLE 1.
Selected eigenvalues for $Re = 10$, $\alpha = 1\text{x}10^{-5}$, $\varepsilon = 1$, $F = S = 0$, $r_2 = 1$.

m_2	c (Asymptotic analysis [6])	c (Numerical, present, $\Delta\bar{T} = 0.01\,K$)
5	$1.33333 + i\,7.53400\text{x}10^{-7}$	$1.33335 + i\,7.53520\text{x}10^{-7}$
20	$2.06020 + i\,1.58950\text{x}10^{-6}$	$2.06017 + i\,1.58909\text{x}10^{-6}$
100	$2.71932 + i\,2.05299\text{x}10^{-5}$	$2.71877 + i\,2.05495\text{x}10^{-5}$

Results and Discussion

Representative stability results are given in Figs. 1 – 4. The viscosity law parameters are chosen for a standard transformer oil-water system (water at the bottom layer). Selecting $\bar{T}_{ref} = \bar{T}_c = 20°C$ gives $(C_1, \bar{d}_1) = (8.836\text{x}10^{-6}, 3411.3\,K)$, $(C_2, \bar{d}_2) = (2.125\text{x}10^{-3}$, 1804.1 $K)$ and thus in all numerical results below $D_2 = 0.53$. However, the remaining parameters are selected in order to reveal their influence on the stability diagrams.

The imaginary part of the complex velocity, $c_i(\alpha)$, is plotted in Fig. 1 for $\Delta\bar{T}$ = 40 K and $\Delta\bar{T}$ = 80 K. Here, $Re = 0.1$, $r_2 = 1$, $F = S = 0$, $\varepsilon = 0.5$, $K_2 = 1$, $m_2 = 0.05$, Pr_1 = 162.5 and Pr_2 = 3.675. Recall that instability is associated with $c_i > 0$. The broken line

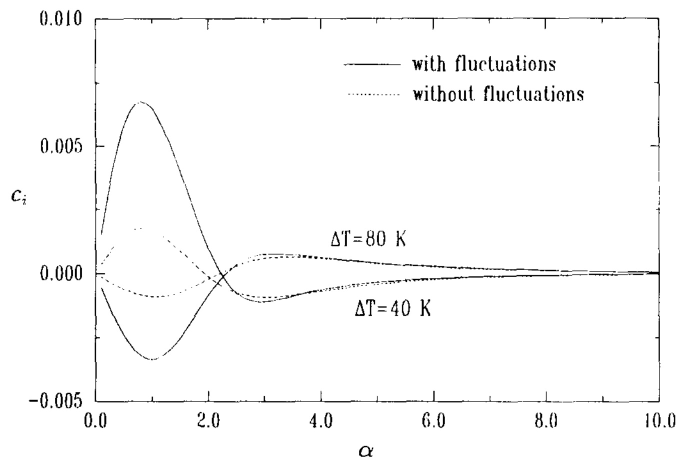

FIG. 1
Imaginary part of complex velocity as a function of wavenumber.
$Re = 0.1$, $r_2 = 1$, $F = S = 0$, $\varepsilon = 0.5$, $K_2 = 1$, $m_2 = 0.05$, $Pr_1 = 162.5$, $Pr_2 = 3.675$, $D_2 = 0.53$.

curves correspond to the simplified formulation in which temperature and viscosity fluctuations are neglected. It is seen that the effect of temperature and viscosity fluctuations is appreciable for $\alpha \lesssim 3$. For larger values of α, their effect diminishes. The effect of temperature and viscosity fluctuations becomes more pronounced for liquids of high Prandtl number as can be seen in Fig. 2

where the imaginary part of the complex velocity c is plotted versus the wavenumber α for various values of Prandtl number and $Re = 0.1$, $r_2 = 1$, $F = S = 0$, $\varepsilon = 0.5$, $K_2 = 1$, $m_2 = 0.05$ and $\Delta\bar{T} = 80\,K$. The case $Pr_1 = Pr_2 = 0$ serves as baseline in the comparison. Note that by setting $Pr_1 = Pr_2 = 0$, temperature and viscosity fluctuations are discarded although terms due to the dependence of viscosity on temperature are retained (see Eqs. (6), (9e) and (7)).

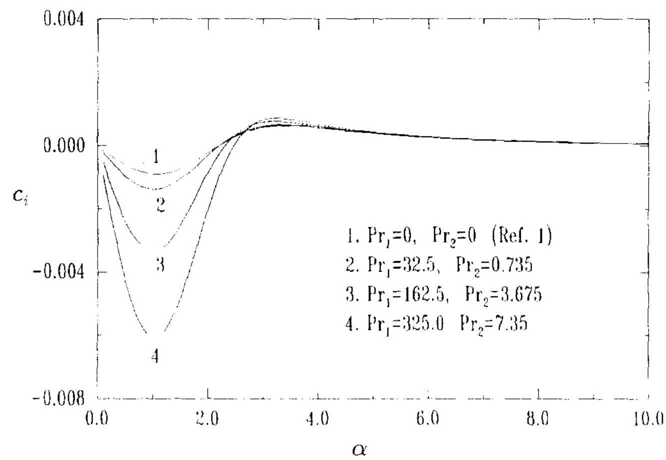

FIG. 2

Imaginary part of complex velocity versus wavenumber for various Prandtl numbers. $Re = 0.1$, $r_2 = 1$, $F = S = 0$, $\varepsilon = 0.5$, $K_2 = 1$, $m_2 = 0.05$, $\Delta\bar{T} = 80\,K$, $D_2 = 0.53$.

In Fig. 3, we present marginal stability curves in the (α - ε) plane for $Re = 0.1$, $r_2 = 1$, $F = S = 0$, $K_2 = 1$, $m_2 = 0.05$, $\Delta\bar{T} = 80\,K$, $Pr_1 = 325$ and $Pr_2 = 7.35$. Fig. 3a corresponds to calculations based on the simplified formulation (temperature and viscosity fluctuations neglected) while Fig. 3b has been constructed based on the full formulation, Eqs. (5)-(9). It is seen that neglecting temperature and viscosity fluctuations can lead to misclassification of the stability of the flow configuration. Fig. 3a indicates that in the range $0.15 \lesssim \varepsilon \lesssim 0.3$ the interface is linearly stable for disturbances of wavenumber $0 < \alpha \lesssim 3$. However, this region of stability is drastically reduced when temperature and viscosity fluctuations are taken into account.

Two-layer, isothermal flows with a very thin layer of the less viscous fluid next to the wall are linearly stable to an interfacial mode [11]. This phenomenon, referred to as the "thin layer effect" or "lubrication stabilization", is predicted by linear stability analysis of disturbances of long ($\alpha \rightarrow 0$) and intermediate wavelengths. Fig. 3 suggests that the thin layer effect is also observed in nonisothermal flows, whether or not viscosity and temperature fluctuations are included in the analysis. The inclusion of viscosity and temperature fluctuations alters the range of thickness ratio

ε for which linearly stable arrangements may exist. Restricting our attention to disturbances of wavenumber $0 < \alpha \leq 10$, Fig. 3a indicates that no disturbance amplification is found until ε reaches approximately 0.04, while Fig. 3b shows that instabilities may arise for values of ε as low as approximately 0.02. Both formulations predict that for a small, fixed value of ε and in the absence of surface tension, interfacial instabilities are more likely to arise first for large values of α. Consequently, $\varepsilon = 0$ may be a vertical asymptote of the stability boundary. However, short wavelength disturbances (large α) are stabilized by surface tension so that (linearly) stable arrangements should exist for small but finite values of ε at least for sufficiently small values of Reynolds number.

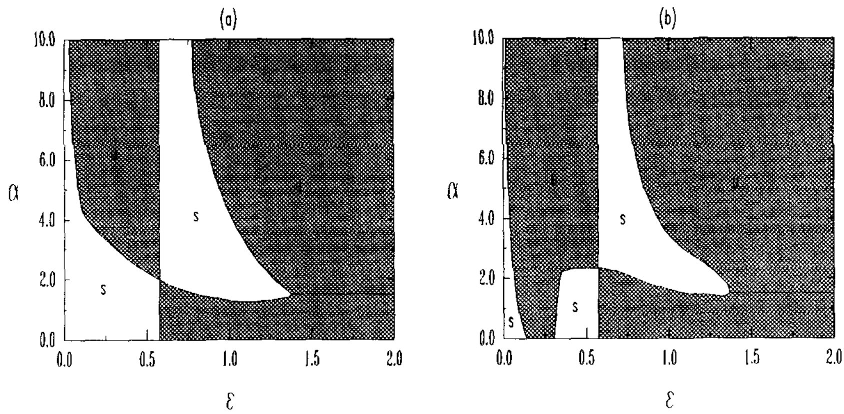

FIG. 3

Marginal stability curves without (map a) and with map (b) temperature and viscosity fluctuations. $Re = 0.1$, $r_2 = 1$, $F = S = 0$, $K_2 = 1$, $m_2 = 0.05$, $\Delta\bar{T} = 80\,K$, $Pr_1 = 325$, $Pr_2 = 7.35$, $D_2 = 0.53$.

Marginal stability curves are shown in Fig. 4 for $Re = 0.1$, $r_2 = 1$, $F = S = 0$, $K_2 = 1$, $m_2 = 0.2$, $\Delta\bar{T} = 50\,K$, $Pr_1 = 325$ and $Pr_2 = 29.4$. In this case, the constant-viscosity ratio m_2 has been increased to 0.2. This corresponds to a smaller jump in the viscosity as one crosses the interface between the less viscous liquid at the bottom layer and the more viscous liquid at the top layer. Again, temperature and viscosity fluctuations are included in Fig. 4b and are neglected in Fig. 4a. It is seen that the stable regions in the $(\alpha - \varepsilon)$ plane contract when temperature and viscosity fluctuations are included in the formulation. The thin layer effect discussed previously in connection with Fig. 3 persists in this case as well. However, the stable region in Fig. 4a obtained for $\varepsilon \lesssim 0.15$ contracts appreciably in Fig. 4b.

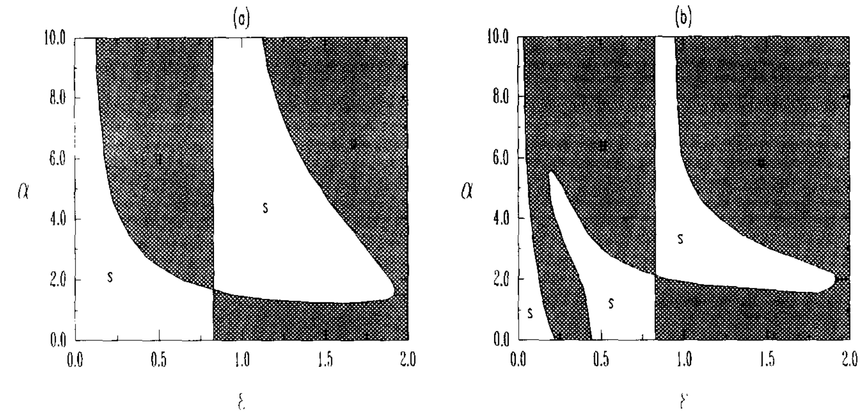

FIG. 4
Marginal stability curves without (map a) and with (map b) temperature and viscosity fluctuations. $Re = 0.1$, $r_2 = 1$, $F = S = 0$, $K_2 = 1$, $m_2 = 0.2$, $\Delta\bar{T} = 50\,K$, $Pr_1 = 325$, $Pr_2 = 29.4$, $D_2 = 0.53$.

To conclude, in contrast to shear instability of single-layer Poiseuille flow (in which the effects of Prandtl number and temperature and viscosity fluctuations are very small), temperature and viscosity fluctuations significantly alter the stability diagrams in multilayer Poiseuille flow especially for large Prandtl number liquids. The feasibility of stabilizing the flow by applying appropriate temperature gradients is under investigation.

Nomenclature

c	complex disturbance velocity
c_p	specific heat at constant pressure
C, d	viscosity law constants
D_2	viscosity exponent constant
k	thermal conductivity
K_2	thermal conductivity ratio
l_1	upper layer thickness
m_2	constant-temperature viscosity ratio
p	pressure
Pr	Prandtl number
r_2	density ratio
t	time
T	temperature
U_o	interface velocity
$\vec{v}$	velocity vector
x, y	cartesian coordinates

Greek Letters

α	disturbance wavenumber
ΔT	wall temperature difference
ε	layer thickness ratio
$\dot{\gamma}$	rate-of-strain tensor
Λ	viscosity disturbance amplitude
μ	fluid viscosity
ϕ	velocity disturbance amplitude
ρ	fluid density
θ	temperature disturbance ampl.

Subscripts/Superscripts

b	base flow
c	cold wall
k	layer number ($k = 1, 2$)
o	reference value
$1, 2$	layer number

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