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Explicit Representations of the Complete Velocity Profile in a Turbulent Boundary Layer

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Nomenclature

B = constant in the logarithmic law
$e(y^+)$ = difference of the approximation, Eq. (10)
$f(y^+)$ = function representing the law of the wall
g = function representing the law of the wake
Re_θ = Reynolds number based on momentum thickness
u = mean velocity component parallel to the wall
u_e = mean velocity at the boundary-layer edge
u_τ = friction velocity, $(\tau_w/\rho)^{1/2}$
u^+ = dimensionless u velocity, u/u_τ
w = wake function
y = coordinate normal to the wall
y^+ = dimensionless distance from the wall, yu_τ/ν
δ, δ^*, θ = boundary-layer, displacement, and momentum thickness, respectively
κ = von Kármán constant
ν = molecular kinematic viscosity
Π = Coles wake parameter
ρ = fluid density
τ_w = wall shear stress

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Introduction

IT is well known that most of the commonly used turbulence models lead to parabolic systems when coupled with the boundary-layer equations. Consequently, starting a differential method of predicting turbulent boundary layers requires the specification of initial profiles for the dependent variables. For this purpose, an accurate and computationally convenient expression for the mean velocity distribution is of particular importance to users of existing computer codes and to developers of new calculation methods. For the problem at hand, a "computationally convenient" formula means a representation of the mean-velocity profile which has some or preferably all of the following characteristics: 1) it is a closed-form expression, 2) it gives u explicitly as a function of y, 3) it is valid over the whole width of the boundary layer, and 4) it is relatively easily evaluated. The objective of this Note is to provide such a formula for external boundary layers and pipe flows. The analysis holds for two-dimensional incompressible turbulent flow past a smooth surface but (being within the framework of the wall-wake similarity laws) fails in the cases of relaxing flows and flows characterized by the presence of very large positive or negative pressure gradients.[1]

Coles[2] has shown that an expression of the form

$$u^+ = \frac{1}{\kappa} \ln y^+ + B + g\left(\Pi, \frac{y}{\delta}\right) \tag{1}$$

provides an accurate fit to experimental velocity data for both equilibrium and nonequilibrium turbulent boundary layers for $y^+ > 50$. However, in order to obtain velocity profiles valid over the whole width of the boundary layer one can write Eq. (1) in a slightly different way

$$u^+ = f(y^+) + g(\Pi, y/\delta) \tag{2}$$

where $f(y^+)$ is a representation of the law of the wall valid over the whole inner layer and is asymptotic at large y^+ to $(1/\kappa) \ln y^+ + B$. Function $g(\Pi, y/\delta)$ is a representation of the law of the wake. In Eqs. (1) and (2), Π is Coles' wake parameter.[1] For the function $g(\Pi, y/\delta)$ we adopt the expression

$$g\left(\Pi, \frac{y}{\delta}\right) = \frac{1}{\kappa}(1+6\Pi)\left(\frac{y}{\delta}\right)^2 - \frac{1}{\kappa}(1+4\Pi)\left(\frac{y}{\delta}\right)^3 \tag{3}$$

proposed independently by Finley et al.,[3] Granville,[4] and Dean.[5] Equation (3) is an improvement over the more widely used form

$$g\left(\Pi, \frac{y}{\delta}\right) = \frac{\Pi}{\kappa} w\left(\frac{y}{\delta}\right) = \frac{2\Pi}{\kappa} \sin^2\left(\frac{\pi}{2}\frac{y}{\delta}\right) \tag{4}$$

since it describes more accurately the true boundary conditions on the wake function.[4] In the present analysis it is assumed that in addition to u_e and ν the parameters u_τ, δ, and Π are known at the streamwise station in question. This is not a stringent requirement since any one of the five parameters u_τ, δ, Π, δ^*, θ can be determined if two of them are known.[1]

Inner Layer (Law of the Wall)

A large variety of analytical representations of the law of the wall have been proposed, characterized by various levels of complexity and accuracy. Spalding's implicit formula may be considered the most widely used and is adopted by White[6] and Dean[5] as the accepted form of the law of the wall.

Spalding[7] has shown that Laufer's experimental data[8] for the mean velocity distribution in the inner layer are well fitted by formulas of the form

$$y^+ = u^+ + e^{-\kappa B}\left[e^{\kappa u^+} - \sum_{n=0}^{n=\ell} \frac{(\kappa u^+)^n}{n!}\right] \tag{5}$$

where $\ell = 3$ or 4 and κ and B are the parameters of the logarithmic law. Equation (5) satisfies (for $\ell = 3$) or exceeds (for $\ell = 4$) Reichardt's cubic power law for the eddy viscosity in the immediate neighborhood of the wall. Furthermore, it is asymptotic at large y^+ to the logarithmic law $u^+ = (1/\kappa)\ln y^+ + B$.

In what follows a very simple procedure of obtaining explicit analytical approximation of Eq. (5) is described. Then, a new formula is proposed which fits better the experimental data. The constants κ and B are taken as 0.41 and 5.0, respectively[1] throughout this work.

Assuming $\ell = 3$ and differentiating Eq. (5) we obtain

$$\frac{dy^+}{du^+} = 1 + \kappa e^{-\kappa B}\left[e^{\kappa u^+} - 1 - (\kappa u^+) - \frac{(\kappa u^+)^2}{2}\right] \tag{6}$$

which by taking into account Eq. (5) can be written as

$$\frac{du^+}{dy^+} = \frac{1}{1 + \kappa\{y^+ - u^+ + e^{-\kappa B}[(\kappa u^+)^3/6]\}} \tag{7}$$

Figure 1 shows a plot of du^+/dy^+ vs y^+. This function can be accurately approximated over the infinite interval $[0,\infty]$ by a rational function of the form

$$R(y^+) = 1 - y^+ \frac{(y^+)^2 + a_1 y^+ + a_0}{(y^+)^3 + b_2 (y^+)^2 + b_1 y^+ + b_0} \tag{8}$$

which correctly emulates the asymptotic behavior of du^+/dy^+ provided that

$$b_2 - a_1 = 1/\kappa = 2.44 \tag{9}$$

Under this condition, the difference

$$e(y^+) = \frac{du^+}{dy^+} - R(y^+) \tag{10}$$

vanishes asymptotically as $y^+ \to \infty$. Moreover, the difference is zero at $y^+ = 0$. The value of the coefficients a_0, a_1, b_0, b_1, b_2 can be conveniently determined by imposing the condition $e(y^+) = 0$ at four points and Eq. (9). A series of numerical experiments have shown that a very good approximation of the function defined by relations (7) and (5) (for $\ell = 3$) is achieved by imposing the condition $e(y^+) = 0$ at $y^+ = (2.0, 6.0, 10.0, 20.0)$. The corresponding values of the coefficients are: $a_0 = 6.0256$, $a_1 = -4.633$, $b_0 = 222.31$, $b_1 = 16.507$, $b_2 = -2.193$.† Analytical integration of Eq. (8) yields

$$u^+ = \ln[(y^+ + 4.67)^{2.24}(y^{+2} - 6.82y^+ + 48.05)^{0.101}] + 4.22\tan^{-1}(0.166y^+ - 0.565) - 1.67 \tag{11}$$

which is in good agreement with Eq. (5). The relative difference is less than 0.8% for $1 < y^+ < 2$ and becomes significantly smaller for larger values of y^+. The above technique has been applied to the case $\ell = 4$. The resulting approximation is

$$u^+ = \ln\left[\frac{(y^+ + 5.85)^{3.04}}{(y^{+2} - 9.25y^+ + 58.5)^{0.3}}\right] + 4.16\tan^{-1}(0.164y^+ - 0.759) - 1.45 \tag{12}$$

The relative difference is less than 0.82% over the interval [1,2], being significantly smaller for larger arguments.

Plots of Eq. (5) for $\ell = 3$ (curve I) and $\ell = 4$ (curve II) are shown in Fig. 2 where comparison is made with the experimental data of Lindgren,[11] Perry,[1] Patel and Head,[12] and Durst.[13] Spalding,[7] based on Laufer's data,[8] was unable to say which of the two curves gives the more precise fit. However, consideration of the experimental data shown in Fig. 2 indicates that curve I gives a better fit for $y^+ < 12$ and curve II shows better agreement for $y^+ > 20$. Thus, a function which closely follows curve I in the immediate neighborhood of the wall turns at $y^+ = 12$ and merges smoothly into curve II in the vicinity of $y^+ = 20$ describes better the data. Following the procedure described above one obtains the following representations of the law of the wall:

$$u^+ = \ln\left[\frac{(y^+ + 11)^{4.02}}{(y^{+2} - 7.37y^+ + 83.3)^{0.79}}\right] + 5.63\tan^{-1}[0.12y^+ - 0.441] - 3.81 \tag{13}$$

The extent to which Eq. (13) fits the data can be judged by inspection of Fig. 2. It should be noted that Eq. (13) is based on no new physical assumptions and correlates data of nonuniform accuracy, obtained from experimental in-

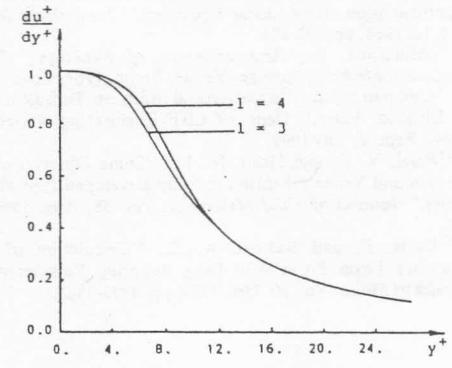

Fig. 1 Dimensionless velocity gradient according to Eqs. (7) and (5).

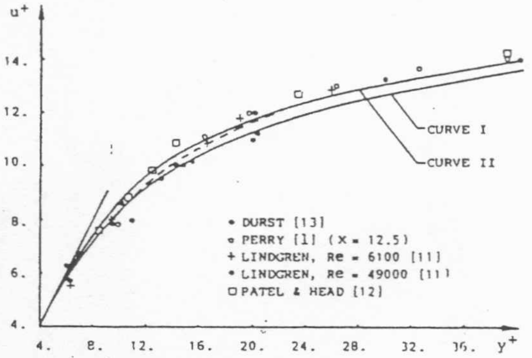

Fig. 2 Comparison of calculated and experimental velocity profiles in the buffer layer. Curve I: Eq. (5), $\ell = 3$. Curve II: Eq. (5), $\ell = 4$. —: Eq. (13).

†It should be noted that if a Remes-type algorithm is employed using the obtained approximation as initial guess it will converge to the best (in the Chebyshev sense) rational approximation.[9,10] However, this is not necessary considering: 1) the very good approximation achieved by the simple interpolation procedure and 2) the nature of the approximated function (function generated by curve fitting of scattered experimental data).

vestigations that have been carried out in different pieces of apparatus.

Conclusions

Any of Eqs. (11), (12), or (13) together with Eq. (3) give an explicit, closed form representation of the mean velocity profile in a turbulent boundary layer which is valid over the whole width of the boundary layer and fits well the experimental data. Equations (11) and (12) are accurate explicit approximations of Spalding's formulas for the law of the wall for $\ell=3$ and $\ell=4$, respectively; while Eq. (13) is a representation of the law of the wall which better fits the experimental data.

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